

# Existence of mild solutions for the Hamilton-Jacobi equation with critical fractional viscosity in the Besov spaces

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## Abstract

We consider the Cauchy problem for the Hamilton-Jacobi equation with critical dissipation,

$$\partial_t u + (-\Delta)^{1/2} u = |\nabla u|^p, \quad x \in \mathbb{R}^N, t > 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,$$

where  $p > 1$  and  $u_0 \in B_{r,1}^1(\mathbb{R}^N) \cap B_{\infty,1}^1(\mathbb{R}^N)$  with  $r \in [1, \infty]$ . We show that for sufficiently small  $u_0 \in \dot{B}_{\infty,1}^1(\mathbb{R}^N)$ , there exists a global-in-time mild solution. Furthermore, we prove that the solution behaves asymptotically like suitable multiplies of the Poisson kernel.

## 1 Introduction

We consider the Hamilton-Jacobi equation with fractional viscosity,

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u = |\nabla u|^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $N \geq 1$ ,  $\partial_t = \partial/\partial t$ ,  $\nabla = (\partial_{x_1}, \dots, \partial_{x_N})$ ,  $\partial_{x_j} = \partial/\partial x_j$  ( $j = 1, \dots, N$ ),  $\alpha \in (0, 2]$ ,  $p > 1$  and  $u_0$  is a nontrivial measurable function in  $\mathbb{R}^N$ . Here the operator  $(-\Delta)^{\alpha/2}$ , which called the Lévy operator, is defined by the Fourier transform  $\mathcal{F}$  such that

$$(-\Delta)^{\alpha/2} f := \mathcal{F}^{-1} [|\xi|^\alpha \mathcal{F}[f]].$$

In this paper we study the existence of global-in-time solutions to the problem (1.1) with  $\alpha = 1$ , and investigate the asymptotics of solutions.

The problem (1.1) with  $\alpha = 2$  is the well-known viscous Hamilton-Jacobi (VHJ) equation. The VHJ equation possesses both mathematical and physical interest. Indeed, in

mathematical points of view, it is the simplest example of a parabolic PDE with a nonlinearity depending only on the first order spatial derivatives of  $u$ , and it describes a model for growing random interfaces, which is known as the Kardar-Parisi-Zhang equation (see [20, 23]). On the other hand, the problem (1.1) with  $\alpha \in (0, 2)$  often appears in the context of mathematical finance as Bellman equations of optimal control of jump diffusion processes (see, for example, [9, 11, 17, 18, 28]).

The VHJ equation has been studied in many papers about various topics. For the existence and uniqueness of solutions, it is well known that, for any  $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ , the problem (1.1) with  $\alpha = 2$  has a unique global-in-time mild solution, i.e., a solution of the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}|\nabla u(\tau)|^p d\tau, \quad t > 0,$$

where  $e^{t\Delta}$  denotes the convolution operator with the heat kernel (see, for example, [2, 4, 6, 10]). Furthermore, this solution is classical for positive time, and by the maximum principle, we see that, if  $u_0 \geq 0$ , then  $u \geq 0$ , and if  $u_0 \leq 0$ , then  $u \leq 0$ . From this property, the nonlinearity  $|\nabla u|^p$  behaves like a source term for nonnegative initial data and an absorption term for nonpositive initial data. Similarly to the case of the semilinear heat equation  $\partial_t u - \Delta u = \lambda|u|^{p-1}u$  with  $\lambda = \pm 1$ , the asymptotics of solutions to this equation is determined by the balance of effects from the diffusion term  $\Delta u$  and the one from the nonlinearity  $|\nabla u|^p$ , and there are many results on the asymptotic behavior of solutions. See, for example, [2]–[6], [10, 19, 24] and the references therein. Among others, in [3], Benachour, Karch and Lanrençot proved that, for the case  $u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$  with  $u_0 \not\equiv 0$ , the following hold.

- (i) Assume that  $u_0 \geq 0$ .
- (a) For the case  $p \geq 2$ , there exists a limit

$$C_* := \lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx + \int_0^\infty \int_{\mathbb{R}^N} |\nabla u(x, t)|^p dx dt \quad (1.2)$$

such that

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{q})+\frac{j}{2}} \|\nabla^j [u(t) - C_* G(t)]\|_{L^q(\mathbb{R}^N)} = 0, \quad q \in [1, \infty], \quad j = 0, 1, \quad (1.3)$$

where  $G(x, t)$  is the heat kernel.

- (b) For the case  $p \in (p_c, 2)$  with  $p_c := (N+2)/(N+1)$ , there exists a positive constant  $\varepsilon = \varepsilon(N, p)$  such that, if

$$\|u_0\|_{L^1(\mathbb{R}^N)} \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}^{(N+1)p-(N+2)} < \varepsilon,$$

then (1.3) holds true.

- (ii) Assume that  $u_0 \leq 0$ . For any  $p > p_c$ , (1.3) holds true.

For the case  $\alpha \in (1, 2)$ , Karch and Woyczyński [19] studied similar topics. They showed that, for any  $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ , the problem (1.1) with  $\alpha \in (1, 2)$  has a unique global-in-time mild solution. Furthermore, for the case  $p > (N + \alpha)/(N + 1)$ , they proved that there exists a (mild) solution which behaves asymptotically like suitable multiples of the kernel of the integral equation. For notions of another weak solutions, Droniou and Imbert [9] constructed a unique global-in-time viscosity solution in  $W^{1,\infty}(\mathbb{R}^N)$  for the case  $\alpha \in (0, 2)$  (see also [11, 27]).

On the other hand, the case  $\alpha = 1$  is completely different from the case  $\alpha \in (1, 2]$ . In fact, for the case  $\alpha \in (0, 2]$ , the semigroup  $e^{-t(-\Delta)^{\alpha/2}}$  satisfies the following decay estimates

$$\|\nabla^j e^{-t(-\Delta)^{\alpha/2}} f\|_{L^q(\mathbb{R}^N)} \leq C t^{-\frac{N}{\alpha}(1-\frac{1}{q})-\frac{j}{\alpha}} \|f\|_{L^1(\mathbb{R}^N)}, \quad q \in [1, \infty], \quad j = 0, 1,$$

for all  $t > 0$  (see, for example, [13]). For the case  $\alpha \in (1, 2]$ , since  $t^{-1/\alpha}$  is integrable locally, we can easily prove the existence of local-in-time mild solutions in  $W^{1,\infty}(\mathbb{R}^N)$  (see [19, Proposition 3.1]). However, for the case  $\alpha = 1$ , since  $t^{-1}$  is not integrable, we need to impose the regularity of one order derivative on the solution. In this sense the value  $\alpha = 1$  is critical. Similar situation appears in the fractional Burgers equation,

$$\partial_t u + u \partial_x u + (-\partial_{xx})^{\alpha/2} u = 0, \quad x \in \mathbb{R}, \quad t > 0. \quad (1.4)$$

For (1.4), the value  $\alpha = 1$  is a threshold for the occurrence of singularity in finite time or the global regularity (see [1, 7, 8, 21]). In [16], the first author of this paper studied (1.4) with  $\alpha = 1$ , and constructed a small global-in-time mild solution in the Besov space  $\dot{B}_{\infty,1}^0(\mathbb{R})$  which is the critical space under the scaling invariance (see also [26]). Furthermore, he proved that, for small initial data in  $L^1(\mathbb{R}) \cap \dot{B}_{\infty,1}^0(\mathbb{R})$ , the corresponding solution behaves like the Poisson kernel as  $t \rightarrow \infty$ .

In this paper, modifying the argument in [16], we show that there exists a global-in-time mild solution of the problem (1.1) with  $\alpha = 1$  in the critical Besov space. Furthermore, we prove that global-in-time solutions with some suitable decay estimates behave asymptotically like suitable multiples of the Poisson kernel.

We introduce some notations. Throughout this paper we put  $\mathcal{L} := -(-\Delta)^{1/2}$  for simplicity. Let  $P_t$  be the Poisson kernel, that is,

$$P_t(x) := t^{-N} P(x/t), \quad x \in \mathbb{R}^N, \quad t > 0,$$

where  $P$  is defined by

$$P(x) := \mathcal{F}^{-1}[e^{-|\xi|}](x) = c_N (1 + |x|^2)^{-(N+1)/2}, \quad x \in \mathbb{R}^N,$$

and  $c_N$  is a constant chosen so that

$$\int_{\mathbb{R}^N} P(x) dx = 1. \quad (1.5)$$

Then, for all  $t > 0$ ,  $e^{t\mathcal{L}}$  denotes the convolution operator with  $P_t$ , that is,

$$[e^{t\mathcal{L}} f](x) := \int_{\mathbb{R}^N} P_t(x - y) f(y) dy, \quad x \in \mathbb{R}^N, \quad t > 0, \quad (1.6)$$

and  $f$  is a measurable function. For  $q \in [1, \infty]$ , we denote by  $\|\cdot\|_{L^q}$  the usual norm of  $L^q := L^q(\mathbb{R}^N)$ . Furthermore, for  $s \in \mathbb{R}$ ,  $q \in [1, \infty]$  and  $\sigma \in (0, \infty]$ , we denote by  $\|\cdot\|_{B_{q,\sigma}^s}$  and  $\|\cdot\|_{\dot{B}_{q,\sigma}^s}$  the usual norm of inhomogeneous and homogeneous Besov spaces  $B_{q,\sigma}^s := B_{q,\sigma}^s(\mathbb{R}^N)$  and  $\dot{B}_{q,\sigma}^s := \dot{B}_{q,\sigma}^s(\mathbb{R}^N)$ , respectively. (See Section 2 for more precise details.)

Now we are ready to state the main result of this paper. We consider the integral equation corresponding to (1.1) with  $\alpha = 1$ , that is,

$$u(t) = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau, \quad t \geq 0, \quad (1.7)$$

and obtain the following result.

**Theorem 1.1** *Let  $N \geq 1$ ,  $p > 1$  and  $r \in [1, \infty]$ . Assume  $u_0 \in B_{q,1}^1$  for all  $q \in [r, \infty]$ . Then the following hold.*

(i) *There exists a positive constant  $\delta = \delta(N, p)$  such that, if*

$$\|u_0\|_{\dot{B}_{\infty,1}^1} \leq \delta, \quad (1.8)$$

*then there exists a unique global-in-time solution  $u$  of (1.7) satisfying*

$$u \in C([0, \infty), B_{q,1}^1) \cap L^1(0, \infty; \dot{B}_{q,1}^2),$$

$$\sup_{t \geq 0} (1+t)^{N(\frac{1}{r}-\frac{1}{q})+j} \|\nabla^j u(t)\|_{L^q} < \infty, \quad (1.9)$$

$$\int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+\frac{1}{p}} \|u(t)\|_{\dot{B}_{q,1}^2} dt < \infty, \quad (1.10)$$

*for all  $q \in [r, \infty]$  and  $j = 0, 1$ .*

(ii) *Let  $v$  be a global-in-time solution of (1.7) satisfying (1.9) and (1.10). Then for any  $j \in \{0, 1\}$ , the following hold.*

(a) *If  $1 < r < \infty$ , then*

$$t^{N(\frac{1}{r}-\frac{1}{q})+j} \|\nabla^j [v(t) - e^{t\mathcal{L}}u_0]\|_{L^q} = \begin{cases} O(t^{-\frac{N}{r}(r-1)}) & \text{if } p \geq r, \\ O(t^{-\frac{N}{r}(p-1)}) & \text{if } p < r, \end{cases} \quad (1.11)$$

*as  $t \rightarrow \infty$ , for any  $q \in [r, \infty]$ .*

(b) *If  $r = 1$ , then the limit  $C_*$  given in (1.2) exists and*

$$\lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j [v(t) - C_* P_{t+1}]\|_{L^q} = 0, \quad 1 \leq q \leq \infty. \quad (1.12)$$

**Remark 1.1** (i) Let  $u$  be a mild solution  $u$  of (1.1) with  $\alpha = 1$ , i.e., solution of (1.7). For any  $\lambda > 0$ , put

$$u_\lambda(x, t) := \lambda^{-1}u(\lambda x, \lambda t), \quad u_{0,\lambda}(x) := \lambda^{-1}u_0(\lambda x). \quad (1.13)$$

Then the function  $u_\lambda$  is also a solution of (1.1) with  $\alpha = 1$  and the initial function  $u_{0,\lambda}$  satisfying

$$C^{-1}\|u_0\|_{\dot{B}_{\infty,1}^1} \leq \|u_{0,\lambda}\|_{\dot{B}_{\infty,1}^1} \leq C\|u_0\|_{\dot{B}_{\infty,1}^1}, \quad (1.14)$$

where  $C$  is a positive constant independent of  $\lambda$ . This means that the condition (1.8) is invariant with respect to the similarity transformation (1.13). This is the reason why we say that  $\dot{B}_{\infty,1}^1$  is the critical Besov space with respect to (1.1).

(ii) In the assertion (ii) of Theorem 1.1, if we only consider the case  $j = 0$ , then we can remove the assumption that the solution  $u$  satisfies (1.10). See Section 5.

(iii) As is seen from our proof, it is possible to replace (1.10) with

$$\int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+\beta} \|u(t)\|_{\dot{B}_{q,1}^2} dt < \infty,$$

where  $1/p < \beta < 1$ . We also note focusing on the linear part that for  $\beta = 1$ , the maximal regularity estimate and the embedding implies that

$$\int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+1} \|e^{t\mathcal{L}}u_0\|_{\dot{B}_{q,1}^2} dt \leq C\|u_0\|_{\dot{B}_{r,1}^0}, \quad B_{r,1}^1 \not\subset \dot{B}_{r,1}^0,$$

and one can not expect the time decay with  $\beta = 1$  for initial data in  $B_{r,1}^1$ . Therefore, the expected maximal decay order is given as the case  $\beta < 1$  except for  $\beta = 1$ , and the case  $\beta = 1/p$  is a sufficient decay to prove the asymptotic behavior.

(iv) By the embedding  $B_{\infty,1}^1 \hookrightarrow C^1$  and  $(-\Delta)^{1/2}f \in C(\mathbb{R}^N)$  for  $f \in B_{\infty,1}^1$ , the solution  $u$  in Theorem 1.1-(i) satisfies the problem (1.1) in the classical sense. We also see that  $u(t)$  is in the class  $C^2$  for almost every  $t$  since  $u \in L^1(0, \infty; \dot{B}_{\infty,1}^2)$ . Compared with the results [9, 11, 27], our framework in the Besov spaces is the one with higher regularity than theirs, since their initial data are in  $W^{1,\infty}$  and solutions are considered in the sense of viscosity solutions and  $B_{\infty,1}^1 \hookrightarrow W^{1,\infty}$ .

(v) In Theorem 1.1-(ii)-(a), it is possible to prove that

$$\lim_{t \rightarrow \infty} t^{N(\frac{1}{r}-\frac{1}{q})+j} \|\nabla^j u(t)\|_{L^q} = 0$$

since one can show  $t^{N(\frac{1}{r}-\frac{1}{q})+j} \|\nabla^j e^{t\mathcal{L}}u_0\|_{L^q} = o(1)$  as  $t \rightarrow \infty$  for any  $u_0 \in L^r$  by the density argument due to  $C_0^\infty \subset L^r$ .

This paper is organized as follows. In Section 2, we give the definition of the Besov spaces, its properties and estimates for the nonlinearity  $|\nabla u|^p$ . We also introduce the linear estimates for  $e^{t\mathcal{L}}f$  in the Lebesgue spaces and the Besov spaces. Sections 3 and 4 are devoted to the proof of the assertions (i) and (ii) in Theorem 1.1, respectively.

## 2 Preliminary

In this section we prove some estimates in the Besov spaces and recall some preliminary results on  $e^{t\mathcal{L}}f$ . In what follows, for any two nonnegative functions  $f_1$  and  $f_2$  on a subset  $D$  of  $[0, \infty)$ , we say

$$f_1(t) \preceq f_2(t), \quad t \in D$$

if there exists a positive constant  $C$  such that  $f_1(t) \leq Cf_2(t)$  for all  $t \in D$ . In addition, we say

$$f_1(t) \asymp f_2(t), \quad t \in D$$

if  $f_1(t) \preceq f_2(t)$  and  $f_2(t) \preceq f_1(t)$  for all  $t \in D$ . We denote the function spaces of rapidly decreasing functions by  $\mathcal{S}(\mathbb{R}^N)$  and tempered distributions by  $\mathcal{S}'(\mathbb{R}^N)$ . We define  $\mathcal{Z}(\mathbb{R}^N)$  by

$$\mathcal{Z}(\mathbb{R}^N) := \left\{ f \in \mathcal{S}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} x^\alpha f(x) dx = 0 \text{ for all } \alpha \in (\{0\} \cup \mathbb{N})^N \right\}$$

with the topology of  $\mathcal{S}(\mathbb{R}^N)$ , and  $\mathcal{Z}'(\mathbb{R}^N)$  by the topological dual of  $\mathcal{Z}(\mathbb{R}^N)$ . We first give the definition of the inhomogeneous and homogeneous Besov spaces (see Triebel [29]).

**Definition 2.1** *Let  $\phi \in \mathcal{S}(\mathbb{R}^N)$  satisfy*

$$\text{supp } \mathcal{F}[\phi] \subset \{ \xi \in \mathbb{R}^N \mid 2^{-1} \leq |\xi| \leq 2 \}, \quad \sum_{j \in \mathbb{Z}} \mathcal{F}[\phi](2^{-j}\xi) = 1 \text{ for any } \xi \in \mathbb{R}^N \setminus \{0\},$$

*Let  $\{\phi_j\}_{j \in \mathbb{Z}}$  and  $\psi$  be defined by*

$$\phi_j(x) := 2^{Nj} \phi(2^j x), \quad \psi(x) = \mathcal{F}^{-1} \left[ 1 - \sum_{j \geq 1} \mathcal{F}[\phi_j] \right] (x)$$

*For  $s \in \mathbb{R}$ ,  $q \in [1, \infty]$  and  $\sigma \in (0, \infty]$ , we define the following.*

(i) *The inhomogeneous Besov space  $B_{q,\sigma}^s$  is defined by*

$$B_{q,\sigma}^s := \left\{ u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{B_{q,\sigma}^s} < \infty \right\},$$

*where*

$$\|u\|_{B_{q,\sigma}^s} := \begin{cases} \|\psi * u\|_{L^q} + \left\{ \sum_{j \geq 1} (2^{js} \|\phi_j * u\|_{L^q})^\sigma \right\}^{1/\sigma} & \text{if } 0 < \sigma < \infty, \\ \|\psi * u\|_{L^q} + \sup_{j \geq 1} 2^{js} \|\phi_j * u\|_{L^q} & \text{if } \sigma = \infty. \end{cases}$$

(ii) *The homogeneous Besov space  $\dot{B}_{q,\sigma}^s$  is defined by*

$$\dot{B}_{q,\sigma}^s := \left\{ u \in \mathcal{Z}'(\mathbb{R}^N) \mid \|u\|_{\dot{B}_{q,\sigma}^s} < \infty \right\},$$

*where*

$$\|u\|_{\dot{B}_{q,\sigma}^s} := \begin{cases} \left\{ \sum_{j \in \mathbb{Z}} (2^{js} \|\phi_j * u\|_{L^q})^\sigma \right\}^{1/\sigma} & \text{if } 0 < \sigma < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\phi_j * u\|_{L^q} & \text{if } \sigma = \infty. \end{cases}$$

**Remark 2.1** It is known that  $\mathcal{Z}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N) \subset \mathcal{Z}'(\mathbb{R}^N)$  and  $\mathcal{Z}'(\mathbb{R}^N) \simeq \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}(\mathbb{R}^N)$ , where  $\mathcal{P}(\mathbb{R}^N)$  is the set of all polynomials, and the homogeneous Besov spaces can also be considered as subspaces of the quotient space  $\mathcal{S}'(\mathbb{R}^N)/\mathcal{P}(\mathbb{R}^N)$ . Then we use the following equivalence, which is due to the argument by e.g. Kozono and Yamazaki [22], for the nonlinear term in (1.1) to construct solutions in the homogeneous spaces with  $u(t) \in \mathcal{S}'(\mathbb{R}^N)$ . If  $s < n/q$  or  $(s, \sigma) = (n/q, 1)$ , then the homogeneous Besov space  $\dot{B}_{q,\sigma}^s$  is regarded as

$$\left\{ u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{\dot{B}_{q,\sigma}^s} < \infty, u = \sum_{j \in \mathbb{Z}} \phi_j * u \text{ in } \mathcal{S}'(\mathbb{R}^N) \right\}.$$

Hence,  $u \in \dot{B}_{q,\sigma}^s$  can be regarded as an element of  $\mathcal{S}'(\mathbb{R}^N)$ . We also see from the analogous argument to theirs that  $\nabla u$  can be regarded as an element of  $\mathcal{S}'(\mathbb{R}^N)$  if  $u \in \dot{B}_{\infty,1}^s$  with  $s \leq 1$ . This is used for the nonlinear term  $|\nabla u|^p$  when we construct global solutions.

Next we give some interpolation inequalities in the Besov spaces.

**Lemma 2.1** Let  $s \in \mathbb{R}$ ,  $\alpha, \beta > 0$ ,  $q \in [1, \infty]$  and  $\sigma \in (0, \infty]$ . Then it holds that

$$\|f\|_{\dot{B}_{q,\sigma}^s} \preceq \|f\|_{\dot{B}_{q,\infty}^{s+\alpha}}^{\frac{\beta}{\alpha+\beta}} \|f\|_{\dot{B}_{q,\infty}^{s-\beta}}^{\frac{\alpha}{\alpha+\beta}} \quad (2.1)$$

for all  $f \in \dot{B}_{q,\infty}^{s+\alpha} \cap \dot{B}_{q,\infty}^{s-\beta}$ .

**Proof.** The estimate (2.1) is known for the case  $1 \leq \sigma \leq \infty$  by the result of Machihara-Ozawa [25]. The case  $0 < \sigma < 1$  follows from the analogous argument to their proof, thus the proof is left to readers.  $\square$

The following proposition is on the equivalence between the norm of the Besov spaces defined as above and that by differences (see Triebel [29]).

**Proposition 2.1** Let  $s > 0$ ,  $q \in [1, \infty]$  and  $\sigma \in (0, \infty]$ . If  $M \in \mathbb{N}$  satisfies  $M > s$ , then there holds that

$$\|f\|_{\dot{B}_{q,\sigma}^s} \asymp \left\{ \int_{\mathbb{R}^N} \left( |\eta|^{-s} \sup_{|y| \leq |\eta|} \|\Delta_y^M f\|_{L^q} \right)^\sigma \frac{d\eta}{|\eta|^N} \right\}^{1/\sigma} \quad (2.2)$$

for all  $f \in \dot{B}_{q,\sigma}^s$ , where  $\Delta_y f(x) := f(x+y) - f(x)$  and  $\Delta_y^M f := (\Delta_y)^M f$ .

By using Proposition 2.1 we have the following.

**Lemma 2.2** Let  $p$ ,  $s$  and  $\varepsilon$  satisfy  $p > 1$ ,  $0 < s < \min\{2, p\}$  and  $0 < \varepsilon < \min\{1, p-1\}$ .

Then, for any  $q \in [1, \infty]$ ,

$$\| |f|^p \|_{\dot{B}_{q,1}^s} \preceq \|f\|_{\dot{B}_{\infty,1}^0}^{p-1} \|f\|_{\dot{B}_{q,1}^s}, \quad (2.3)$$

$$\begin{aligned} & \| |f|^p - |g|^p \|_{\dot{B}_{\infty,1}^\varepsilon} \\ & \preceq (\|f\|_{\dot{B}_{\infty,1}^0}^{p-1} + \|g\|_{\dot{B}_{\infty,1}^0}^{p-1}) \|f - g\|_{\dot{B}_{q,1}^\varepsilon} \\ & + \begin{cases} \left( \|f\|_{\dot{B}_{\infty,1}^0}^{p-1-\varepsilon} \|f\|_{\dot{B}_{\infty,1}^1}^\varepsilon + \|g\|_{\dot{B}_{\infty,1}^0}^{p-1-\varepsilon} \|g\|_{\dot{B}_{\infty,1}^1}^\varepsilon \right) \|f - g\|_{\dot{B}_{q,1}^0}, & \text{if } 1 < p < 2, \\ \left( \|f\|_{\dot{B}_{\infty,1}^0}^{p-2} + \|g\|_{\dot{B}_{\infty,1}^0}^{p-2} \right) \left( \|f\|_{\dot{B}_{\infty,1}^0}^{1-\varepsilon} \|f\|_{\dot{B}_{\infty,1}^1}^\varepsilon + \|g\|_{\dot{B}_{\infty,1}^0}^{1-\varepsilon} \|g\|_{\dot{B}_{\infty,1}^1}^\varepsilon \right) \|f - g\|_{\dot{B}_{q,1}^0}, & \text{if } p \geq 2, \end{cases} \end{aligned} \quad (2.4)$$

for all  $f, g \in \dot{B}_{\infty,1}^0 \cap \dot{B}_{\infty,1}^1 \cap \dot{B}_{q,1}^0 \cap \dot{B}_{q,1}^{\max\{s,\varepsilon\}}$ .

In order to prove this lemma, we use the following fundamental inequality.

**Lemma 2.3** *Let  $p > 1$ . Then, for any  $A, B, C, D \in \mathbb{R}$ ,*

$$\begin{aligned} & \left| |A|^p - |B|^p - (|C|^p - |D|^p) \right| \\ & \preceq (\|C\|^{p-1} + \|D\|^{p-1}) |A - B - (C - D)| \\ & + \begin{cases} (|A - C|^{p-1} + |B - D|^{p-1}) |A - B|, & \text{if } 1 < p < 2, \\ (|A|^{p-2} + |B|^{p-2} + |C|^{p-2} + |D|^{p-2}) (|A - C| + |C - D|) |A - B|, & \text{if } p \geq 2. \end{cases} \end{aligned} \quad (2.5)$$

**Proof.** Let  $p > 1$  and  $A, B, C, D \in \mathbb{R}$ . It follows from the fundamental theorem of calculus that

$$\begin{aligned} & |A|^p - |B|^p - (|C|^p - |D|^p) \\ & = \int_0^1 \left\{ \partial_\theta |\theta A + (1 - \theta) B|^p - \partial_\theta |\theta C + (1 - \theta) D|^p \right\} d\theta \\ & = \int_0^1 \left\{ |\theta A + (1 - \theta) B|^{p-2} (\theta A + (1 - \theta) B) (A - B) \right. \\ & \quad \left. - |\theta C + (1 - \theta) D|^{p-2} (\theta C + (1 - \theta) D) (C - D) \right\} d\theta \\ & = \int_0^1 \left[ \left\{ |\theta A + (1 - \theta) B|^{p-2} (\theta A + (1 - \theta) B) \right. \right. \\ & \quad \left. \left. - |\theta C + (1 - \theta) D|^{p-2} (\theta C + (1 - \theta) D) \right\} (A - B) \right. \\ & \quad \left. + |\theta C + (1 - \theta) D|^{p-2} (\theta C + (1 - \theta) D) (A - B - (C - D)) \right] d\theta. \end{aligned} \quad (2.6)$$

Furthermore, we have

$$\left| |E|^{p-2} E - |F|^{p-2} F \right| \leq C \begin{cases} |E - F|^{p-1}, & \text{if } 1 < p < 2, \\ (|E|^{p-2} + |F|^{p-2}) |E - F|, & \text{if } p \geq 2, \end{cases}$$



for any  $E, F \in \mathbb{R}$ . This together with (2.6) yields (2.5). Thus Lemma 2.3 follows.  $\square$

**Proof of Lemma 2.2.** For the proof of (2.3), we utilize the equivalent norm (2.2) of the Besov spaces  $\dot{B}_{q,1}^s$  by differences, and it suffices to estimate the following

$$\int_{\mathbb{R}^N} \left( |\eta|^{-s} \sup_{|y| \leq |\eta|} \|\Delta_y^2 |f|^p\|_{L^q} \right) \frac{d\eta}{|\eta|^N}.$$

In order to estimate  $\Delta_y^2 |f|^p$ , we apply Lemma 2.2. Put

$$A = f(x + 2y), \quad B = C = f(x + y), \quad D = f(x),$$

and we note that

$$A - B = (\Delta_y f)(x + y), \quad C - D = (\Delta_y f)(x), \quad A - B - (C - D) = (\Delta_y^2 f)(x).$$

In the case  $1 < p < 2$ , by (2.5) and the Hölder inequality we have

$$\begin{aligned} & \|\Delta_y^2 |f|^p\|_{L^q} \\ & \leq \|f\|_{L^\infty}^{p-1} \|\Delta_y^2 f\|_{L^q} + (\| |\Delta_y f(\cdot + y)|^{p-1} \|_{L^{\frac{pq}{p-1}}} + \| |\Delta_y f|^{p-1} \|_{L^{\frac{pq}{p-1}}}) \|\Delta_y f\|_{L^{pq}} \\ & \leq \|f\|_{L^\infty}^{p-1} \|\Delta_y^2 f\|_{L^q} + \|\Delta_y f\|_{L^{pq}}^p \\ & =: I_1(y) + I_2(y). \end{aligned} \tag{2.7}$$

On the estimate of  $I_1$ , we get

$$\begin{aligned} & \int_{\mathbb{R}^N} |\eta|^{-s} \sup_{|y| \leq |\eta|} I_1(y) \frac{d\eta}{|\eta|^N} \\ & \leq \|f\|_{L^\infty}^{p-1} \int_{\mathbb{R}^N} \left( |\eta|^{-s} \sup_{|y| \leq |\eta|} \|\Delta_y^2 f\|_{L^q} \right) \frac{d\eta}{|\eta|^N} \leq \|f\|_{\dot{B}_{\infty,1}^0}^{p-1} \|f\|_{\dot{B}_{q,1}^s}. \end{aligned} \tag{2.8}$$

On the estimate of  $I_2$ , applying the Hölder inequality and the embedding  $\dot{B}_{\infty,1}^0 \hookrightarrow \dot{B}_{\infty,\infty}^0$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |\eta|^{-s} \sup_{|y| \leq |\eta|} I_2(y) \frac{d\eta}{|\eta|^N} = \int_{\mathbb{R}^N} \left( |\eta|^{-\frac{s}{p}} \sup_{|y| \leq |\eta|} \|\Delta_y f\|_{L^{pq}} \right)^p \frac{d\eta}{|\eta|^N} \\ & \leq \|f\|_{\dot{B}_{pq,p}^{\frac{s}{p}}}^p \leq \left( \|f\|_{\dot{B}_{\infty,\infty}^0}^{1-\frac{1}{p}} \|f\|_{\dot{B}_{q,1}^s}^{\frac{1}{p}} \right)^p \leq \|f\|_{\dot{B}_{\infty,1}^0}^{p-1} \|f\|_{\dot{B}_{q,1}^s}. \end{aligned} \tag{2.9}$$

In the case  $p \geq 2$ , by (2.5) and the Hölder inequality again we obtain

$$\begin{aligned} & \|\Delta_y^2 f\|_{L^q} \leq \|f\|_{L^\infty}^{p-1} \|\Delta_y^2 f\|_{L^q} + \|f\|_{L^\infty}^{p-2} (\|\Delta_y f(\cdot + y)\|_{L^{2q}} + \|\Delta_y f\|_{L^{2q}}) \\ & \quad \times (\|\Delta_y f(\cdot + y)\|_{L^{2q}} + \|\Delta_y f\|_{L^{2q}}) \\ & \leq \|f\|_{L^\infty}^{p-1} \|\Delta_y^2 f\|_{L^q} + \|f\|_{L^\infty}^{p-2} \|\Delta_y f\|_{L^{2q}}^2 \\ & =: I_1(y) + I_3(y). \end{aligned} \tag{2.10}$$

For the estimate of  $I_3$ , it follows from the same estimate as (2.9) with taking  $p = 2$  for  $\|\Delta_y f\|_{L^{2q}}$  that

$$\int_{\mathbb{R}^N} |\eta|^{-s} \sup_{|y| \leq |\eta|} I_3(y) \frac{d\eta}{|\eta|^N} \preceq \|f\|_{\dot{B}_{\infty,1}^{p-2}}^{p-2} \left( \|f\|_{\dot{B}_{\infty,1}^0} \|f\|_{\dot{B}_{q,1}^s} \right) = \|f\|_{\dot{B}_{\infty,1}^0}^{p-1} \|f\|_{\dot{B}_{q,1}^s}. \quad (2.11)$$

Combining (2.7), (2.8), (2.9), (2.10) and (2.11), the estimate (2.3) holds.

For the proof of (2.4), we also utilize the equivalent norm (2.2) of the Besov space  $\dot{B}_{\infty,1}^\varepsilon$  by differences, and it suffices to estimate the following

$$\int_{\mathbb{R}^N} \left( |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} \|\Delta_y(|f|^p - |g|^p)\|_{L^q} \right) \frac{d\eta}{|\eta|^N}.$$

In order to estimate  $\Delta_y(|f|^p - |g|^p)$ , we also apply Lemma 2.2. Put

$$A = f(x+y), \quad B = g(x+y), \quad C = f(x), \quad D = g(x),$$

we note that

$$A - C = (\Delta_y f)(x), \quad B - D = (\Delta_y g)(x), \quad A - B - (C - D) = \Delta_y(f - g).$$

In the case  $1 < p < 2$ , by (2.5) and the Hölder inequality we have

$$\begin{aligned} \|\Delta_y(|f|^p - |g|^p)\|_{L^q} &\preceq (\|f\|_{L^\infty}^{p-1} + \|g\|_{L^\infty}^{p-1}) \|\Delta_y(f - g)\|_{L^q} \\ &\quad + (\|\Delta_y f\|_{L^\infty}^{p-1} + \|\Delta_y g\|_{L^\infty}^{p-1}) \|f - g\|_{L^q} \\ &=: J_1(y) + J_2(y). \end{aligned} \quad (2.12)$$

On the estimate of  $J_1$ , we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_1(y) \frac{d\eta}{|\eta|^N} &\preceq (\|f\|_{L^\infty}^{p-1} + \|g\|_{L^\infty}^{p-1}) \int_{\mathbb{R}^N} \left( |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} \|\Delta_y(f - g)\|_{L^q} \right) \frac{d\eta}{|\eta|^N} \\ &\preceq (\|f\|_{\dot{B}_{\infty,1}^0}^{p-1} + \|g\|_{\dot{B}_{\infty,1}^0}^{p-1}) \|f - g\|_{\dot{B}_{q,1}^\varepsilon}. \end{aligned} \quad (2.13)$$

On the estimate of  $J_2$ , by (2.1) and the embeddings  $\dot{B}_{q,1}^s \hookrightarrow \dot{B}_{q,\infty}^s$  ( $s = 0, 1$ ) and  $\dot{B}_{q,1}^0 \hookrightarrow L^q$  we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_2(y) \frac{d\eta}{|\eta|^N} \\ &\preceq \int_{\mathbb{R}^N} \left( |\eta|^{-\frac{\varepsilon}{p-1}} \sup_{|y| \leq |\eta|} (\|\Delta_y f\|_{L^\infty} + \|\Delta_y g\|_{L^\infty}) \right)^{p-1} \frac{d\eta}{|\eta|^N} \|f - g\|_{L^q} \\ &\preceq \left( \|f\|_{\dot{B}_{\infty,p-1}^{\frac{p-1}{p-1}}}^{p-1} + \|g\|_{\dot{B}_{\infty,p-1}^{\frac{p-1}{p-1}}}^{p-1} \right) \|f - g\|_{L^q} \\ &\preceq \left\{ \left( \|f\|_{\dot{B}_{\infty,\infty}^0}^{1-\frac{\varepsilon}{p-1}} \|f\|_{\dot{B}_{\infty,\infty}^1}^{\frac{\varepsilon}{p-1}} \right)^{p-1} + \left( \|g\|_{\dot{B}_{\infty,\infty}^0}^{1-\frac{\varepsilon}{p-1}} \|g\|_{\dot{B}_{\infty,\infty}^1}^{\frac{\varepsilon}{p-1}} \right)^{p-1} \right\} \|f - g\|_{L^q} \\ &\preceq \left( \|f\|_{\dot{B}_{\infty,1}^0}^{p-1-\varepsilon} \|f\|_{\dot{B}_{\infty,1}^1}^\varepsilon + \|g\|_{\dot{B}_{\infty,1}^0}^{p-1-\varepsilon} \|g\|_{\dot{B}_{\infty,1}^1}^\varepsilon \right) \|f - g\|_{\dot{B}_{q,1}^0}. \end{aligned} \quad (2.14)$$

In the case  $p \geq 2$ , by (2.5) and the Hölder inequality again we obtain

$$\begin{aligned} \|\Delta_y(|f|^p - |g|^p)\|_{L^q} &\preceq (\|f\|_{L^\infty}^{p-1} + \|g\|_{L^\infty}^{p-1}) \|\Delta_y(f - g)\|_{L^q} \\ &\quad + (\|f\|_{L^\infty}^{p-2} + \|g\|_{L^\infty}^{p-2}) (\|\Delta_y f\|_{L^\infty} + \|\Delta_y g\|_{L^\infty}) \|f - g\|_{L^q} \\ &=: J_1(y) + J_3(y). \end{aligned} \quad (2.15)$$

For the estimate of  $J_3$ , it follows from (2.1) and the embeddings  $\dot{B}_{q,1}^s \hookrightarrow \dot{B}_{q,\infty}^s$  ( $s = 0, 1$ ) and  $\dot{B}_{q,1}^0 \hookrightarrow L^q$  that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_3(y) \frac{d\eta}{|\eta|^N} \\ &\preceq (\|f\|_{L^\infty}^{p-2} + \|g\|_{L^\infty}^{p-2}) \int_{\mathbb{R}^N} \left( |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} (\|\Delta_y f\|_{L^\infty} + \|\Delta_y g\|_{L^\infty}) \right) \frac{d\eta}{|\eta|^N} \|f - g\|_{L^q} \\ &\preceq (\|f\|_{L^\infty}^{p-2} + \|g\|_{L^\infty}^{p-2}) (\|f\|_{\dot{B}_{\infty,1}^\varepsilon} + \|g\|_{\dot{B}_{\infty,1}^\varepsilon}) \|f - g\|_{L^q} \\ &\preceq (\|f\|_{\dot{B}_{\infty,1}^0}^{p-2} + \|g\|_{\dot{B}_{\infty,1}^0}^{p-2}) (\|f\|_{\dot{B}_{\infty,1}^{1-\varepsilon}} \|f\|_{\dot{B}_{\infty,1}^\varepsilon} + \|g\|_{\dot{B}_{\infty,1}^{1-\varepsilon}} \|g\|_{\dot{B}_{\infty,1}^\varepsilon}) \|f - g\|_{\dot{B}_{q,1}^0}. \end{aligned} \quad (2.16)$$

Therefore, (2.4) is obtained by (2.12), (2.13), (2.14), (2.15) and (2.16).  $\square$

The end of this section we recall some results on  $e^{t\mathcal{L}}f$ .

**Lemma 2.4** [13, 16] *Let  $s \in \mathbb{R}$  and  $q \in [1, \infty]$ .*

(i) *For  $j = 0, 1$ ,  $\alpha \geq 0$ ,  $r \in [1, q]$  and  $\sigma \in [1, \infty]$ , it holds that*

$$\|\nabla^j P_{t+1}\|_{L^q} \preceq (1+t)^{-N(1-\frac{1}{q})-j}, \quad (2.17)$$

$$\|\nabla^j e^{t\mathcal{L}}f\|_{L^q} \preceq t^{-N(\frac{1}{r}-\frac{1}{q})-j} \|f\|_{L^r}, \quad (2.18)$$

$$\|e^{t\mathcal{L}}f\|_{\dot{B}_{q,\sigma}^{s+\alpha}} \preceq t^{-N(\frac{1}{r}-\frac{1}{q})-\alpha} \|f\|_{\dot{B}_{r,\sigma}^s}. \quad (2.19)$$

(ii) *It holds that*

$$\|e^{t\mathcal{L}}f\|_{L_t^1(0,\infty;\dot{B}_{q,1}^{s+1})} \preceq \|f\|_{\dot{B}_{q,1}^s}. \quad (2.20)$$

(iii) *It holds that*

$$\left\| \int_0^t e^{(t-\tau)\mathcal{L}} f(\tau) d\tau \right\|_{L_t^1(0,\infty;\dot{B}_{q,1}^{s+1})} \preceq \|f\|_{L^1(0,\infty;\dot{B}_{q,1}^s)}. \quad (2.21)$$

**Lemma 2.5** ([Proposition 3.1][14]). *For any  $f \in L^1$ , if*

$$\int_{\mathbb{R}^N} f(x) dx = 0,$$

*then*

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{L}}f\|_{L^1} = 0.$$

### 3 Existence of global-in-time solutions and Decay estimates

In this section we prove the assertion (i) of Theorem 1.1. We apply the contraction mapping principle in a suitable complete metric space. Let  $\Psi(u)$  be defined by

$$\Psi(u)(t) := e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau, \quad (3.1)$$

and we define the following norms

$$\begin{aligned} \|u\|_{\dot{X}_q^s} &:= \sup_{t>0} \|u(t)\|_{\dot{B}_{q,1}^s} + \int_0^\infty \|u(t)\|_{\dot{B}_{q,1}^{s+1}} dt, \\ \|u\|_{\dot{Y}_q^s} &:= \|u\|_{\dot{X}_q^s} + \sup_{t>0} t^{N(\frac{1}{r}-\frac{1}{q})+1-s} \|u(t)\|_{\dot{B}_{q,1}^1} + \int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+1-s} \|u(t)\|_{\dot{B}_{q,1}^2} dt. \end{aligned}$$

$\dot{X}_q^s$  with the norm  $\|u\|_{\dot{X}_q^s}$  is defined by the space of all functions  $u$  such that

$$u \in L^\infty(0, \infty; \dot{B}_{q,1}^s) \cap L^1(0, \infty; \dot{B}_{q,1}^{s+1}) \quad \text{and} \quad \|u\|_{\dot{X}_q^s} < \infty,$$

and  $\dot{Y}_q^s$  with the norm  $\|\cdot\|_{\dot{Y}_q^s}$  is also defined by the space of all functions  $u$  such that

$$u \in \dot{X}_q^s \quad \text{and} \quad \|u\|_{\dot{Y}_q^s} < \infty.$$

Here, let  $\varepsilon$  and  $\lambda$  be fixed constants satisfying

$$0 < \varepsilon < \min\{1, p-1\} \quad \text{and} \quad 0 < \lambda < 1, \quad (3.2)$$

and we introduce the following metric space  $\mathfrak{X}$

$$\begin{aligned} \mathfrak{X} := \{ & u \in \dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon \cap \dot{X}_r^1 \cap \dot{X}_\infty^1 \mid \|u\|_{\dot{X}_q^1} \leq 2C_0 \|u_0\|_{\dot{B}_{q,1}^1} \text{ for any } q \in [r, \infty], \\ & \|u\|_{\dot{Y}_r^\lambda \cap \dot{Y}_\infty^\lambda} \leq 2C_0 \|u_0\|_{\dot{B}_{r,1}^\lambda \cap \dot{B}_{\infty,1}^\lambda} \}, \end{aligned}$$

with the metric

$$d(u, v) := \|u - v\|_{\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon},$$

where  $C_0$  will be taken later. We first show that  $\mathfrak{X}$  is a complete metric space.

**Lemma 3.1**  *$\mathfrak{X}$  is a complete metric space.*

**Proof.** It is easy to see that  $\mathfrak{X}$  is a metric space, then we show the completeness only. Let  $\{u_n\}$  be a Cauchy sequence in  $\mathfrak{X}$ . Since  $\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon$  is complete, there exists  $u \in \dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon$  such that  $u_n$  converges to  $u$  in  $\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon$  as  $n \rightarrow \infty$ . Then we also have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\phi_j * (u_n(t) - u(t))\|_{L^r \cap L^\infty} &= 0 \quad \text{for almost every } t \text{ and } j \in \mathbb{Z}, \\ \lim_{n \rightarrow \infty} \int_0^L \|\phi_j * (u_n(t) - u(t))\|_{L^r \cap L^\infty} dt &= 0 \quad \text{for any } L > 0 \text{ and } j \in \mathbb{Z}. \end{aligned}$$

There imply that, for any  $L > 0$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{|j| \leq L} 2^j \|\phi_j * u_n(t)\|_{L^r \cap L^\infty} &= \sum_{|j| \leq L} 2^j \|\phi_j * u(t)\|_{L^r \cap L^\infty}, \\ \lim_{n \rightarrow \infty} t^{N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda} \sum_{|j| \leq L} 2^j \|\phi_j * u_n(t)\|_{L^q} &= t^{N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda} \sum_{|j| \leq L} 2^j \|\phi_j * u(t)\|_{L^q} \end{aligned}$$

for almost every  $t$ , and

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^L \sum_{|j| \leq L} 2^{2j} \|\phi_j * u_n(t)\|_{L^r \cap L^\infty} dt &= \int_0^L \sum_{|j| \leq L} 2^{2j} \|\phi_j * u(t)\|_{L^r \cap L^\infty} dt, \\ \lim_{n \rightarrow \infty} \int_0^L t^{N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda} \sum_{|j| \leq L} 2^{2j} \|\phi_j * u_n(t)\|_{L^r \cap L^\infty} dt \\ &= \int_0^L t^{N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda} \sum_{|j| \leq L} 2^{2j} \|\phi_j * u(t)\|_{L^r \cap L^\infty} dt. \end{aligned}$$

The terms in right hand side of the above four equalities are bounded uniformly with respect to  $L > 0$  since  $\{u_n\} \subset \mathfrak{X}$ , and they are monotone increasing, so that, they converges as  $L \rightarrow \infty$ . Then we deduce that  $u$  satisfies

$$\|u\|_{\dot{X}_q^1} \leq 2C_0 \|u_0\|_{\dot{B}_{q,1}^1} \quad \text{for any } q \in [r, \infty] \quad \text{and} \quad \|u\|_{\dot{Y}_r^\lambda \cap \dot{Y}_\infty^\lambda} \leq 2C_0 \|u_0\|_{\dot{B}_{r,1}^\lambda \cap \dot{B}_{\infty,1}^\lambda},$$

hence,  $u \in \mathfrak{X}$ . Therefore the completeness of  $\mathfrak{X}$  follows.  $\square$

In order to estimate the terms in (3.1), we prepare the following proposition.

**Proposition 3.1** *Let  $p, q, r, \varepsilon$  and  $\lambda$  satisfy  $p > 1$ ,  $1 \leq r \leq q \leq \infty$  and (3.2). Then there holds that*

$$\|e^{t\mathcal{L}} u_0\|_{\dot{X}_q^1} \preceq \|u_0\|_{\dot{B}_{q,1}^1}, \quad \|e^{t\mathcal{L}} u_0\|_{\dot{Y}_q^\lambda} \preceq \|u_0\|_{\dot{B}_{q,1}^\lambda} + \|u_0\|_{\dot{B}_{r,1}^\lambda}, \quad (3.3)$$

$$\left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{L^q} \preceq t \|u\|_{\dot{X}_\infty^1}^{p-1} \|u\|_{\dot{X}_q^1}, \quad (3.4)$$

$$\left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{\dot{X}_q^1} \preceq \|u\|_{\dot{X}_\infty^1}^{p-1} \|u\|_{\dot{X}_q^1}, \quad (3.5)$$

$$\left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{\dot{Y}_q^\lambda} \preceq \|u\|_{\dot{X}_\infty^1}^{p-1} (\|u\|_{\dot{Y}_r^\lambda} + \|u\|_{\dot{Y}_q^\lambda}), \quad (3.6)$$

$$\left\| \int_0^t e^{(t-\tau)\mathcal{L}} (|\nabla u(\tau)|^p - |\nabla v(\tau)|^p) d\tau \right\|_{\dot{X}_q^\varepsilon} \preceq (\|u\|_{\dot{X}_\infty^1}^{p-1} + \|v\|_{\dot{X}_\infty^1}^{p-1}) \|u - v\|_{\dot{X}_q^\varepsilon}. \quad (3.7)$$

**Remark 3.1** *We should note that the nonlinear term  $|\nabla u(\tau)|^p$  is in  $\mathcal{S}'(\mathbb{R}^N)$  if  $u \in \dot{X}_\infty^1$ . Although  $\dot{B}_{\infty,1}^1(\mathbb{R}^N)$  is considered as a subspace of  $\mathcal{Z}'(\mathbb{R}^N)$ ,  $\nabla u(\tau)$  is determined independently of the choice of representative elements in  $\dot{B}_{\infty,1}^1$  by  $\nabla u(\tau) \in \dot{B}_{\infty,1}^0(\mathbb{R}^N)$  and Remark 2.1, hence,  $\nabla u(\tau) \in \mathcal{S}'(\mathbb{R}^N)$ . We also see  $|\nabla u(\tau)|^p \in L^\infty(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N)$  by  $\dot{B}_{\infty,1}^0 \subset L^\infty$ . In addition the estimate (3.4) implies that the term in the left hand side is in  $L^q$  if  $u \in \mathfrak{X}$ .*

**Proof.** The linear estimate (3.3) is verified by the use of (2.19) and (2.20). In fact, the estimates of  $\|e^{t\mathcal{L}}u_0\|_{\dot{X}_q^s}$  ( $s = 1, \lambda$ ) is obtained by the boundedness of  $e^{t\mathcal{L}}$ , (2.19) and (2.20), and for the second and third terms in the definition of  $\|\cdot\|_{\dot{Y}_q^\lambda}$  we also apply (2.19) and (2.20) to get

$$\begin{aligned} t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda}\|e^{t\mathcal{L}}u_0\|_{\dot{B}_{q,1}^1} &\preceq \|u_0\|_{\dot{B}_{r,1}^\lambda}, \\ \int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda}\|e^{t\mathcal{L}}u_0\|_{\dot{B}_{q,1}^2} dt &\preceq \int_0^\infty \|e^{\frac{t}{2}\mathcal{L}}u_0\|_{\dot{B}_{r,1}^{1+\lambda}} dt \preceq \|u_0\|_{\dot{B}_{r,1}^\lambda}. \end{aligned}$$

Then (3.3) is obtained.

The estimate (3.4) is obtained by applying the boundedness of  $e^{(t-\tau)\mathcal{L}}$  from  $L^q$  to itself, the Hölder inequality and the embedding  $\dot{B}_{q,1}^0 \hookrightarrow L^q$ . Thus we omit the detail.

To prove the nonlinear estimates (3.5), (3.6) and (3.7), we prepare the following nonlinear estimates that, for  $s = 1, \lambda$ ,

$$\| |\nabla u|^p \|_{\dot{B}_{q,1}^s} \preceq \| \nabla u \|_{\dot{B}_{\infty,1}^0}^{p-1} \| \nabla u \|_{\dot{B}_{q,1}^s} \preceq \| u \|_{\dot{B}_{\infty,1}^1}^{p-1} \| u \|_{\dot{B}_{q,1}^{s+1}}, \quad (3.8)$$

$$\begin{aligned} &\| |\nabla u|^p - |\nabla v|^p \|_{\dot{B}_{q,1}^\varepsilon} \\ &\preceq (\| \nabla u \|_{\dot{B}_{\infty,1}^0}^{p-1} + \| \nabla v \|_{\dot{B}_{\infty,1}^0}^{p-1}) \| \nabla u - \nabla v \|_{\dot{B}_{q,1}^\varepsilon} \\ &\quad + \left( \| \nabla u \|_{\dot{B}_{\infty,1}^0}^{p-1-\varepsilon} \| \nabla u \|_{\dot{B}_{\infty,1}^1}^\varepsilon + \| \nabla v \|_{\dot{B}_{\infty,1}^0}^{p-1-\varepsilon} \| \nabla v \|_{\dot{B}_{\infty,1}^1}^\varepsilon \right) \| \nabla u - \nabla v \|_{\dot{B}_{q,1}^0} \\ &\preceq (\| u \|_{\dot{B}_{\infty,1}^1}^{p-1} + \| v \|_{\dot{B}_{\infty,1}^1}^{p-1}) \| u - v \|_{\dot{B}_{q,1}^{1+\varepsilon}} \\ &\quad + \| u - v \|_{\dot{B}_{q,1}^\varepsilon} \| u - v \|_{\dot{B}_{q,1}^{1+\varepsilon}}^{1-\varepsilon} \times \\ &\quad \times \begin{cases} \left( \| u \|_{\dot{B}_{\infty,1}^1}^{p-1-\varepsilon} \| u \|_{\dot{B}_{\infty,1}^2}^\varepsilon + \| v \|_{\dot{B}_{\infty,1}^1}^{p-1-\varepsilon} \| v \|_{\dot{B}_{\infty,1}^2}^\varepsilon \right), & \text{if } 1 < p < 2, \\ \left( \| u \|_{\dot{B}_{\infty,1}^1}^{p-2} + \| v \|_{\dot{B}_{\infty,1}^1}^{p-2} \right) \left( \| u \|_{\dot{B}_{\infty,1}^1}^{1-\varepsilon} \| u \|_{\dot{B}_{\infty,1}^2}^\varepsilon + \| v \|_{\dot{B}_{\infty,1}^1}^{1-\varepsilon} \| v \|_{\dot{B}_{\infty,1}^2}^\varepsilon \right), & \text{if } p \geq 2, \end{cases} \end{aligned} \quad (3.9)$$

which are obtained by (2.3), (2.4) and the interpolation inequality in the Besov spaces, that is,

$$\| f \|_{\dot{B}_{q,1}^1} = \sum_{j \in \mathbb{Z}} 2^j \| \phi_j * f \|_{L^q} = \sum_{j \in \mathbb{Z}} (2^{\varepsilon j})^\varepsilon (2^{(1+\varepsilon)j})^{1-\varepsilon} \| \phi_j * f \|_{L^q}^\varepsilon \| \phi_j * f \|_{L^q}^{1-\varepsilon} \leq \| f \|_{\dot{B}_{q,1}^\varepsilon}^\varepsilon \| f \|_{\dot{B}_{q,1}^{1+\varepsilon}}^{1-\varepsilon}.$$

On the estimate of (3.5), by the boundedness of  $e^{(t-\tau)\mathcal{L}}$ , (2.21), (3.8) with  $s = 1$  and the Hölder inequality we have

$$\begin{aligned} \left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{\dot{X}_q^1} &\preceq \int_0^\infty \| |\nabla u(\tau)|^p \|_{\dot{B}_{q,1}^1} d\tau \\ &\preceq \| u \|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1} \| u \|_{L^1(0,\infty;\dot{B}_{q,1}^2)} \\ &\preceq \| u \|_{\dot{X}_\infty^1}^{p-1} \| u \|_{\dot{X}_q^1}. \end{aligned}$$

Then (3.5) is obtained.

On the estimate of (3.6), the first norm  $\|\cdot\|_{\dot{X}_q^\lambda}$  in the definition of  $\|\cdot\|_{\dot{Y}_q^\lambda}$  can be treated in the same way as the proof of (3.5) with (3.8) ( $s = \lambda$ ), thus we omit the estimate on  $\|\cdot\|_{\dot{X}_q^\lambda}$  to consider the second and third terms only. We put

$$K_1(x, t) := \int_0^{t/2} e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau, \quad K_2(x, t) := \int_{t/2}^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau, \quad (3.10)$$

for all  $x \in \mathbb{R}^N$  and  $t > 0$ . On the second term, by (2.19) and (3.8) we have

$$\begin{aligned} & t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|K_1(t)\|_{\dot{B}_{q,1}^1} \\ & \preceq t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \int_0^{t/2} (t-\tau)^{-N(\frac{1}{r}-\frac{1}{q})-1+\lambda} \| |\nabla u(\tau)|^p \|_{\dot{B}_{r,1}^\lambda} d\tau \\ & \preceq \int_0^{t/2} \|u(\tau)\|_{\dot{B}_{\infty,1}^1}^{p-1} \|u(\tau)\|_{\dot{B}_{r,1}^{\lambda+1}} d\tau \\ & \preceq \|u\|_{\dot{X}_\infty^1}^{p-1} \|u\|_{\dot{Y}_r^\lambda}, \quad t > 0. \end{aligned} \quad (3.11)$$

Furthermore, by the boundedness of  $e^{(t-\tau)\mathcal{L}}$  and (3.8) we obtain

$$\begin{aligned} & t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|K_2(t)\|_{\dot{B}_{q,1}^1} \\ & \preceq t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \int_{t/2}^t \| |\nabla u(\tau)|^p \|_{\dot{B}_{q,1}^1} d\tau \\ & \preceq \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)} \int_{t/2}^t \tau^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|u(\tau)\|_{\dot{B}_{q,1}^2} d\tau \\ & \preceq \|u\|_{\dot{X}_\infty^1}^{p-1} \|u\|_{\dot{Y}_q^\lambda}, \quad t > 0. \end{aligned} \quad (3.12)$$

On the third norm, by (2.19), (2.20) and (3.8) we have

$$\begin{aligned} & \int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|K_1(t)\|_{\dot{B}_{q,1}^2} dt \\ & \preceq \int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \int_0^{t/2} (t-\tau)^{-N(\frac{1}{r}-\frac{1}{q})-1+\lambda} \|e^{\frac{t-\tau}{2}\mathcal{L}} |\nabla u(\tau)|^p\|_{\dot{B}_{r,1}^{\lambda+1}} d\tau dt \\ & \preceq \int_0^\infty \int_{2\tau}^\infty \|e^{\frac{t-\tau}{2}\mathcal{L}} |\nabla u(\tau)|^p\|_{\dot{B}_{r,1}^{\lambda+1}} dt d\tau \preceq \int_0^\infty \| |\nabla u(\tau)|^p \|_{\dot{B}_{r,1}^\lambda} d\tau \\ & \preceq \|u\|_{\dot{X}_\infty^1}^{p-1} \|u\|_{\dot{Y}_r^\lambda}. \end{aligned} \quad (3.13)$$

Furthermore, by (2.20) and (3.8) we obtain

$$\begin{aligned}
& \int_0^\infty t^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|K_2(t)\|_{\dot{B}_{q,1}^2} dt \\
& \preceq \int_0^\infty \tau^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \int_\tau^{2\tau} \|e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p\|_{\dot{B}_{q,1}^2} dt d\tau \\
& \preceq \int_0^\infty \tau^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \| |\nabla u(\tau)|^p \|_{\dot{B}_{q,1}^1} d\tau \\
& \preceq \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1} \int_0^\infty \tau^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|u(\tau)\|_{\dot{B}_{q,1}^2} d\tau \\
& \preceq \|u\|_{\dot{X}_\infty^1}^{p-1} \|u\|_{\dot{Y}_q^\lambda}.
\end{aligned} \tag{3.14}$$

Then, (3.6) is obtained by (3.10), (3.11), (3.12), (3.13) and (3.14).

For the proof of (3.7), it follows from the boundedness of  $e^{(t-\tau)\mathcal{L}}$ , (2.21), (3.9) and the Hölder inequality that, if  $1 < p < 2$ , then

$$\begin{aligned}
& \left\| \int_0^t e^{(t-\tau)\mathcal{L}} (|\nabla u(\tau)|^p - |\nabla v(\tau)|^p) d\tau \right\|_{\dot{X}_q^\varepsilon} \\
& \preceq \int_0^\infty \| |\nabla u(\tau)|^p - |\nabla v(\tau)|^p \|_{\dot{B}_{q,1}^\varepsilon} d\tau \\
& \preceq \left( \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1} + \|v\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1} \right) \|u - v\|_{L^1(0,\infty;\dot{B}_{q,1}^{1+\varepsilon})} \\
& \quad + \|u - v\|_{L^\infty(0,\infty;\dot{B}_{q,1}^\varepsilon)}^\varepsilon \|u - v\|_{L^1(0,\infty;\dot{B}_{q,1}^{1+\varepsilon})}^{1-\varepsilon} \times \\
& \quad \times \left( \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1-\varepsilon} \|u\|_{L^1(0,\infty;\dot{B}_{\infty,1}^2)}^\varepsilon + \|v\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1-\varepsilon} \|v\|_{L^1(0,\infty;\dot{B}_{\infty,1}^2)}^\varepsilon \right) \\
& \preceq \left( \|u\|_{\dot{X}_\infty^1}^{p-1} + \|v\|_{\dot{X}_\infty^1}^{p-1} \right) \|u - v\|_{\dot{X}_q^\varepsilon},
\end{aligned}$$

and, if  $p \geq 2$ , then

$$\begin{aligned}
& \left\| \int_0^t e^{(t-\tau)\mathcal{L}} (|\nabla u(\tau)|^p - |\nabla v(\tau)|^p) d\tau \right\|_{\dot{X}_q^\varepsilon} \\
& \preceq \left( \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1} + \|v\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-1} \right) \|u - v\|_{L^1(0,\infty;\dot{B}_{q,1}^{1+\varepsilon})} \\
& \quad + \|u - v\|_{L^\infty(0,\infty;\dot{B}_{q,1}^\varepsilon)}^\varepsilon \|u - v\|_{L^1(0,\infty;\dot{B}_{q,1}^{1+\varepsilon})}^{1-\varepsilon} \left( \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-2} + \|v\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{p-2} \right) \times \\
& \quad \times \left( \|u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{1-\varepsilon} \|u\|_{L^1(0,\infty;\dot{B}_{\infty,1}^2)}^\varepsilon + \|v\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^1)}^{1-\varepsilon} \|v\|_{L^1(0,\infty;\dot{B}_{\infty,1}^2)}^\varepsilon \right) \\
& \preceq \left( \|u\|_{\dot{X}_\infty^1}^{p-1} + \|v\|_{\dot{X}_\infty^1}^{p-1} \right) \|u - v\|_{\dot{X}_q^\varepsilon},
\end{aligned}$$

Therefore, (3.7) is obtained and the proof of all estimates is completed.  $\square$

In what follows, we prove that the solution exists globally in time by applying the contraction mapping principle in  $\mathfrak{X}$  for initial data  $u_0$  in  $B_{r,1}^1 \cap B_{\infty,1}^1$  and small in  $\dot{B}_{\infty,1}^1$ , and that the solutions satisfy the decay estimates (1.9) and (1.10).



**Proof of existence of global-in-time solutions in  $\mathfrak{X} \cap C([0, \infty), B_{r,1}^1 \cap B_{\infty,1}^1)$ .** Let the constant  $C_0$  in the definition of  $\mathfrak{X}$  be a constant which satisfy the all estimates in Proposition 3.1, and we assume the initial data satisfies

$$u_0 \in B_{r,1}^1 \cap B_{\infty,1}^1 \quad \text{and} \quad \|u_0\|_{\dot{B}_{\infty,1}^1} \leq (2^{p+1}C_0^p)^{-\frac{1}{p-1}}. \quad (3.15)$$

For any  $u, v \in \mathfrak{X}$ , it follows from Proposition 3.1 that

$$\begin{aligned} \|\Psi(u)\|_{\dot{X}_q^s} &\leq C_0\|u_0\|_{\dot{B}_{q,1}^s} + C_0\|u\|_{\dot{X}_\infty^{p-1}}\|u\|_{\dot{X}_q^s} \\ &\leq C_0\|u_0\|_{\dot{B}_{q,1}^s} + C_0(2C_0\|u_0\|_{\dot{B}_{\infty,1}^1})^{p-1} \cdot 2C_0\|u_0\|_{\dot{B}_{q,1}^s} \\ &\leq 2C_0\|u_0\|_{\dot{B}_{q,1}^s}, \\ \|\Psi(u)\|_{\dot{Y}_r^\lambda \cap \dot{Y}_\infty^\lambda} &\leq C_0\|u_0\|_{\dot{B}_{r,1}^\lambda \cap \dot{B}_{\infty,1}^\lambda} + C_0\|u\|_{\dot{X}_\infty^{p-1}}\|u\|_{\dot{Y}_r^\lambda} \\ &\leq C_0\|u_0\|_{\dot{B}_{r,1}^\lambda \cap \dot{B}_{\infty,1}^\lambda} + C_0(2C_0\|u_0\|_{\dot{B}_{\infty,1}^1})^{p-1} \cdot 2C_0\|u_0\|_{\dot{B}_{r,1}^\lambda \cap \dot{B}_{\infty,1}^\lambda} \\ &\leq 2C_0\|u_0\|_{\dot{B}_{r,1}^\lambda \cap \dot{B}_{\infty,1}^\lambda}, \\ \|\Psi(u) - \Psi(v)\|_{\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon} &\leq C_0(\|u\|_{\dot{X}_\infty^{p-1}} + \|v\|_{\dot{X}_\infty^{p-1}})\|u - v\|_{\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon} \\ &\leq C_0 \cdot 2(2C_0\|u_0\|_{\dot{B}_{\infty,1}^1})^{p-1} \cdot \|u - v\|_{\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon} \\ &\leq \frac{1}{2}\|u - v\|_{\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon}. \end{aligned} \quad (3.16)$$

for any  $s = \varepsilon, 1$ .  $\Psi$  is a contraction map from  $\mathfrak{X}$  to itself and the global solution for small initial data is obtained in  $\mathfrak{X}$ . Then  $u(t) = \Psi(u)(t)$  in  $\mathcal{Z}'(\mathbb{R}^N)$  for almost every  $t$ , and we have to find a fixed point such that the equality  $u(t) = \Psi(u)(t)$  holds in  $\mathcal{S}'(\mathbb{R}^N)$ . For this purpose, we take a sequence  $\{u_n\}$  such that

$$u_1 := e^{t\mathcal{L}}u_0, \quad u_n := \Psi(u_{n-1}), \quad n \geq 2.$$

The previous contraction argument implies that  $u_n$  converges to  $u$  in  $\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon$ . Here, we see that  $\Psi(u_{n-1})(t)$  tends to  $\Psi(u)(t)$  in  $L^\infty$  as  $n \rightarrow \infty$  for each  $t$  since we have from (3.4)

$$\begin{aligned} \|\Psi(u_{n-1})(t)\|_{L^\infty} &\preceq \|u_0\|_{L^\infty} + t\|u_{n-1}\|_{\dot{X}_\infty^1}^p, \quad \|\Psi(u)(t)\|_{L^\infty} \preceq \|u_0\|_{L^\infty} + t\|u\|_{\dot{X}_\infty^1}^p, \\ \left\| \int_0^t e^{(t-\tau)\mathcal{L}} (|\nabla u_{n-1}(\tau)|^p - |\nabla u(\tau)|^p) d\tau \right\|_{L^\infty} \\ &\preceq \int_0^t \left( \|\nabla u_{n-1}\|_{L^\infty}^{p-1} + \|\nabla u(\tau)\|_{L^\infty}^{p-1} \right) \|\nabla u_{n-1}(\tau) - \nabla u(\tau)\|_{L^\infty} d\tau \\ &\preceq \left( \|u_{n-1}\|_{\dot{X}_\infty^{p-1}}^{p-1} + \|u\|_{\dot{X}_\infty^{p-1}}^{p-1} \right) \int_0^t \|u_{n-1}(\tau) - u(\tau)\|_{\dot{B}_{\infty,1}^1} d\tau \\ &\preceq \|u_0\|_{\dot{B}_{\infty,1}^1}^{p-1} \int_0^t \|u_{n-1}(\tau) - u(\tau)\|_{\dot{B}_{\infty,1}^\varepsilon} \|u_{n-1}(\tau) - u(\tau)\|_{\dot{B}_{\infty,1}^{1+\varepsilon}}^{1-\varepsilon} d\tau \\ &\preceq \|u_0\|_{\dot{B}_{\infty,1}^1}^{p-1} t^\varepsilon \|u_{n-1} - u\|_{L^\infty(0,\infty;\dot{B}_{\infty,1}^\varepsilon)}^\varepsilon \|u_{n-1} - u\|_{L^1(0,\infty;\dot{B}_{\infty,1}^{1+\varepsilon})}^{1-\varepsilon} \\ &\preceq \|u_0\|_{\dot{B}_{\infty,1}^1}^{p-1} t^\varepsilon \|u_{n-1} - u\|_{\dot{X}_\infty^\varepsilon} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore,  $u_n(t) = \Psi(u_{n-1})(t)$  is a Cauchy sequence in  $L^\infty$ , so that, there exists  $v(t) \in L^\infty$  such that  $u_n(t)$  converges to  $v(t)$  in  $L^\infty$  as  $n \rightarrow \infty$ . It follows from  $L^\infty \subset \mathcal{S}'(\mathbb{R}^N) \subset \mathcal{Z}'(\mathbb{R}^N)$  and the uniqueness of the limit in  $\mathcal{Z}'(\mathbb{R}^N)$  that  $u_n(t)$  also converges to  $v(t)$  in  $\mathcal{Z}'(\mathbb{R}^N)$  as  $n \rightarrow \infty$  and  $v(t) = u(t)$  in  $\mathcal{Z}'(\mathbb{R}^N)$ . Since  $u(t) \in \dot{B}_{\infty,1}^1$  and  $\nabla u(t) \in \mathcal{S}'(\mathbb{R}^N)$  by Remark 2.1, it holds that  $\nabla v(t) = \nabla u(t)$  and  $\Psi(v)(t) = \Psi(u)(t)$  in  $\mathcal{S}'(\mathbb{R}^N)$  for all  $t$ . Then taking the limit in the topology of  $L^\infty$  on the equation  $u_n(t) = \Psi(u_{n-1})(t)$  for each  $t$ , we obtain

$$v(t) = \Psi(u)(t) = \Psi(v)(t) \quad \text{in } L^\infty.$$

By  $\|e^{t\mathcal{L}}u_0\|_{L^q} \leq \|u_0\|_{L^q}$  and (3.4),  $v$  satisfies  $v(t) \in L^r \cap \dot{B}_{r,1}^1 \cap L^\infty \cap \dot{B}_{\infty,1}^1 = B_{r,1}^1 \cap B_{\infty,1}^1$ . Hence, the fixed point  $v$  is a solution in  $\mathfrak{X} \cap C([0, \infty), B_{r,1}^1 \cap B_{\infty,1}^1)$ .

It remains to show the uniqueness. Let  $u, v \in \mathfrak{X}$  satisfies  $u, v \in C([0, \infty), B_{r,1}^1 \cap B_{\infty,1}^1)$ ,  $u = \Psi(u)$  and  $v = \Psi(v)$ , and we show that  $u(t) = v(t)$  in  $\mathcal{S}'(\mathbb{R}^N)$  for all  $t$ . The contraction property (3.16) implies that  $u(t) = v(t)$  in  $\dot{B}_{\infty,1}^\varepsilon \subset \mathcal{Z}'(\mathbb{R}^N)$ . Since  $0 < \varepsilon < 1$ , there exists a constant  $c(t)$  independent of  $x \in \mathbb{R}^N$  such that  $u(t) = v(t) + c(t)$  in  $\mathcal{S}'(\mathbb{R}^N)$ . It follows from  $\nabla u(t) = \nabla v(t)$  in  $\mathcal{S}'(\mathbb{R}^N)$  that

$$u(t) = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}|\nabla u(\tau)|^p d\tau = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}|\nabla v(\tau)|^p d\tau = v(t) \quad \text{in } \mathcal{S}'(\mathbb{R}^N).$$

Therefore,  $c(t) \equiv 0$  in  $\mathcal{S}'(\mathbb{R}^N)$ , and the uniqueness follows.  $\square$

**Proof of the decay estimates (1.9) and (1.10).** According to the above proof of global existence, let

$$\lambda = (p-1)/p, \tag{3.17}$$

and it is sufficient to show only (1.9) for the solution  $u$  satisfying

$$\|u\|_{\dot{X}_r^1 \cap \dot{X}_\infty^1 \cap \dot{Y}_r^\lambda \cap \dot{Y}_\infty^\lambda} < \infty, \tag{3.18}$$

since (1.10) is obtained by  $\|u\|_{\dot{Y}_r^\lambda \cap \dot{Y}_\infty^\lambda} < \infty$ . In the case  $0 \leq t \leq 1$ , the boundedness in time on the norms  $\|\nabla^j u(t)\|_{L^q}$  ( $j = 0, 1$ ) is obtained by the Hölder inequality, the inequalities  $\|u(t)\|_{L^q} \preceq \|u(t)\|_{\dot{B}_{q,1}^0}$ ,  $\|\nabla u(t)\|_{L^q} \preceq \|u(t)\|_{\dot{B}_{q,1}^1}$ , and (3.18), so that it suffices to consider the case  $t > 1$ . We show the estimate with derivative:

$$\|\nabla u(t)\|_{L^q} \preceq (1+t)^{-N(\frac{1}{r}-\frac{1}{q})-1}, \quad t > 0. \tag{3.19}$$

Once (3.19) is proved, it is possible to show the decay estimate of  $\|u(t)\|_{L^q}$ . In fact, by (2.18) and (3.19) we see that

$$\begin{aligned} & \|u(t)\|_{L^q} \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})} \|u_0\|_{L^r} + \int_0^{t/2} (t-\tau)^{-N(\frac{1}{r}-\frac{1}{q})} \|\nabla u(\tau)\|_{L^r}^p d\tau + \int_{t/2}^t \|\nabla u(\tau)\|_{L^q}^p d\tau \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})} + t^{-N(\frac{1}{r}-\frac{1}{q})} \int_0^{t/2} \|\nabla u(\tau)\|_{L^{pr}}^p d\tau + \int_{t/2}^t \|\nabla u(\tau)\|_{L^{pq}}^p d\tau \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})} + t^{-N(\frac{1}{r}-\frac{1}{q})} \int_0^{t/2} \{(1+\tau)^{-N(\frac{1}{r}-\frac{1}{pr})-1}\}^p d\tau + \int_{t/2}^t \{\tau^{-N(\frac{1}{r}-\frac{1}{pq})-1}\}^p d\tau \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})}, \quad t \geq 1, \end{aligned}$$

so that, the decay estimate in  $L^q(\mathbb{R}^n)$  is obtained.

We show (3.19). It follows from (2.18) that

$$\|\nabla u(t)\|_{L^q} \leq t^{-N(\frac{1}{r}-\frac{1}{q})-1} \|u_0\|_{L^r} + \left( \int_0^{t/2} + \int_{t/2}^t \right) \|\nabla e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p\|_{L^q} d\tau. \quad (3.20)$$

We first consider the case  $r < \infty$ . By (2.18), (3.17) and (3.18) we have

$$\begin{aligned} \int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p\|_{L^q} d\tau &\leq \int_0^{t/2} (t-\tau)^{-N(\frac{1}{r}-\frac{1}{q})-1} \|\nabla u(\tau)\|_{L^r}^p d\tau \\ &\leq t^{-N(\frac{1}{r}-\frac{1}{q})-1} \int_0^{t/2} \|\nabla u(\tau)\|_{L^{pr}}^p d\tau \\ &\leq t^{-N(\frac{1}{r}-\frac{1}{q})-1} \int_0^{t/2} \{(1+\tau)^{-N(\frac{1}{r}-\frac{1}{pr})-1+\lambda}\}^p d\tau \\ &\leq t^{-N(\frac{1}{r}-\frac{1}{q})-1}. \end{aligned} \quad (3.21)$$

On the other hand, by (3.17), (3.18), the boundedness of  $e^{(t-s)\mathcal{L}}$  in  $L^q$  and the Hölder inequality we obtain

$$\begin{aligned} &\int_{t/2}^t \|\nabla e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p\|_{L^q} d\tau \\ &\leq \int_{t/2}^t \|\nabla |\nabla u(\tau)|^p\|_{L^q} d\tau \\ &\leq \int_{t/2}^t \|u(\tau)\|_{\dot{B}_{\infty,1}^{p-1}}^{p-1} \|u(\tau)\|_{\dot{B}_{q,1}^2} d\tau \\ &\leq \|u\|_{\dot{Y}_{\infty}^{\lambda}}^{p-1} \int_{t/2}^t (\tau^{-\frac{N}{r}-1+\lambda})^{p-1} \tau^{-N(\frac{1}{r}-\frac{1}{q})-1+\lambda} \tau^{N(\frac{1}{r}-\frac{1}{q})+1-\lambda} \|u(\tau)\|_{\dot{B}_{q,1}^2} d\tau \\ &\leq (t^{-\frac{N}{r}-1+\lambda})^{p-1} t^{-N(\frac{1}{r}-\frac{1}{q})-1+\lambda} \|u\|_{\dot{Y}_{\infty}^{\lambda}}^{p-1} \|u\|_{\dot{Y}_q^{\lambda}} \\ &\leq t^{-N(\frac{1}{r}-\frac{1}{q})-1}. \end{aligned} \quad (3.22)$$

This together with (3.20) and (3.21) yields (3.19) for the case  $r < \infty$ .

Next we consider the case  $r = \infty$ . In this case, the problem is that the integral in the third line of (3.21) diverges as  $t \rightarrow \infty$ . Then corresponding estimate to (3.21) is the following with taking  $r = q = \infty$

$$\int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p\|_{L^\infty} d\tau \leq t^{-1} \int_0^{t/2} \{(1+\tau)^{-1+\lambda}\}^p d\tau \leq t^{-1} \log(1+t),$$

and the same estimate as (3.22) holds. This implies that

$$\|\nabla u(t)\|_{L^\infty} \leq t^{-1} \log(1+t), \quad t > 1.$$

By this decay estimate, we can improve the corresponding one to (3.21) as

$$\int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p\|_{L^\infty} d\tau \leq t^{-1} \int_0^{t/2} \{(1+\tau)^{-1} \log(2+\tau)\}^p d\tau \leq t^{-1}.$$

Therefore, we also have the estimate (3.19) for the case  $r = \infty$ , and the proof of (1.9) is completed.  $\square$

## 4 Asymptotic behavior

In this section we prove the assertion (ii) of Theorem 1.1. The proof is based on the arguments in [14, Theorem 1.2] and [15, Theorem 1.1] (see, also, [12]). Throughout this section we assume that  $u$  is a global-in-time solution of (1.7) satisfying (1.9) and (1.10).

**Proof of Theorem 1.1 (ii)-(a).** Let  $r \in (1, \infty)$  and  $q \in [r, \infty]$ . By (1.7), for any  $j \in \{0, 1\}$ , we have

$$\begin{aligned} & \|\nabla^j[v(t) - e^{t\mathcal{L}}u_0]\|_{L^q} \\ & \leq \left\| \nabla^j \int_{t/2}^t e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p d\tau \right\|_{L^q} + \left\| \nabla^j \int_0^{t/2} e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p d\tau \right\|_{L^q} \\ & =: I_{1,j}(t) + I_{2,j}(t) \end{aligned} \quad (4.1)$$

for all  $t > 0$ . We first estimate  $I_{1,j}$ . By (1.9) and (2.18) we obtain

$$\begin{aligned} I_{1,0}(t) & \leq \int_{t/2}^t \|e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p\|_{L^q} d\tau \\ & \leq \int_{t/2}^t \|\nabla v(\tau)\|_{L^\infty}^{p-1} \|\nabla v(\tau)\|_{L^q} d\tau \\ & \preceq \int_{t/2}^t \tau^{-(\frac{N}{r}+1)(p-1)} \tau^{-N(\frac{1}{r}-\frac{1}{q})-1} d\tau \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})-(\frac{N}{r}+1)(p-1)}, \quad t \geq 1. \end{aligned} \quad (4.2)$$

Furthermore, applying the argument similar to (3.22) with (1.9) and (1.10), we see that

$$\begin{aligned} I_{1,1}(t) & \leq \int_{t/2}^t \|\nabla e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p\|_{L^q} d\tau \\ & \leq \int_{t/2}^t \|\nabla v(\tau)\|_{L^\infty}^{p-1} \|\nabla^2 v(\tau)\|_{L^q} d\tau \\ & \preceq \int_{t/2}^t \tau^{-(\frac{N}{r}+1)(p-1)} \tau^{-N(\frac{1}{r}-\frac{1}{q})-\frac{1}{p}} \tau^{N(\frac{1}{r}-\frac{1}{q})+\frac{1}{p}} \|\nabla^2 v(\tau)\|_{L^q} d\tau \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})-\frac{1}{p}-(\frac{N}{r}+1)(p-1)} \int_0^\infty \tau^{N(\frac{1}{r}-\frac{1}{q})+\frac{1}{p}} \|v(\tau)\|_{\dot{B}_{q,1}^2} d\tau \\ & \preceq t^{-N(\frac{1}{r}-\frac{1}{q})-\frac{1}{p}-(\frac{N}{r}+1)(p-1)}, \quad t \geq 1. \end{aligned} \quad (4.3)$$

Since  $(p-1)p+1 > p$  for all  $p > 1$ , it follows from (4.2) and (4.3) that

$$t^{N(\frac{1}{r}-\frac{1}{q})+j} I_{1,j}(t) = O(t^{-\frac{N}{r}(p-1)}) \quad (4.4)$$

as  $t \rightarrow \infty$ , for any  $j \in \{0, 1\}$ .

Next we estimate  $I_{2,j}$ . For the case  $1 < r \leq p$ , by (1.9) and (2.18) we obtain

$$\begin{aligned}
I_{2,j}(t) &\leq \int_0^{t/2} \|\nabla^j e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p\|_{L^q} d\tau \\
&\preceq \int_0^{t/2} (t-\tau)^{-N(1-\frac{1}{q})-j} \| |\nabla v(\tau)|^p \|_{L^1} d\tau \\
&\preceq t^{-N(1-\frac{1}{q})-j} \int_0^{t/2} \|\nabla v(\tau)\|_{L^p}^p d\tau \\
&\preceq t^{-N(\frac{1}{r}-\frac{1}{q})-j-\frac{N}{r}(r-1)} \int_0^\infty (1+\tau)^{-N(\frac{p}{r}-1)-p} d\tau \\
&\preceq t^{-N(\frac{1}{r}-\frac{1}{q})-j-\frac{N}{r}(r-1)}, \quad t \geq 1.
\end{aligned} \tag{4.5}$$

For the case  $r > p$ , by (1.9) and (2.18) again we see that

$$\begin{aligned}
I_{2,j}(t) &\leq \int_0^{t/2} \|\nabla^j e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p\|_{L^q} d\tau \\
&\preceq \int_0^{t/2} (t-\tau)^{-N(\frac{p}{r}-\frac{1}{q})-j} \| |\nabla v(\tau)|^p \|_{L^{\frac{r}{p}}} d\tau \\
&\preceq t^{-N(\frac{p}{r}-\frac{1}{q})-j} \int_0^{t/2} \|\nabla v(\tau)\|_{L^r}^p d\tau \\
&\preceq t^{-N(\frac{1}{r}-\frac{1}{q})-j-\frac{N}{r}(p-1)} \int_0^\infty (1+\tau)^{-p} d\tau \preceq t^{-N(\frac{1}{r}-\frac{1}{q})-j-\frac{N}{r}(p-1)}, \quad t \geq 1.
\end{aligned}$$

This together with (4.5) yields

$$t^{N(\frac{1}{r}-\frac{1}{q})-j} I_{2,j}(t) = \begin{cases} O(t^{-\frac{N}{r}(r-1)}) & \text{if } p \geq r, \\ O(t^{-\frac{N}{r}(p-1)}) & \text{if } p < r, \end{cases} \tag{4.6}$$

as  $t \rightarrow \infty$ , for any  $j \in \{0, 1\}$ . Therefore, substituting (4.4) and (4.6) into (4.1), we have (1.11), and the assertion (ii)-(a) of Theorem 1.1 follows.  $\square$

**Proof of Theorem 1.1 (ii)-(b).** Let  $r = 1$ . Put

$$c(t) := M(u_0) + \int_0^t M(|\nabla v(\tau)|^p) d\tau, \tag{4.7}$$

where

$$M(f) = \int_{\mathbb{R}^N} f(x) dx. \tag{4.8}$$

Then, by (1.9) we have

$$|c(t_2) - c(t_1)| = \int_{t_1}^{t_2} M(|\nabla v(\tau)|^p) d\tau = \int_{t_1}^{t_2} \|\nabla v(\tau)\|_{L^p}^p d\tau \preceq \int_{t_1}^{t_2} (1+\tau)^{-N(p-1)-p} d\tau$$

for all  $t_2 \geq t_1 \geq 0$ . This implies that there exists the limit  $C_*$  given by (1.2) such that

$$\left| \int_{\mathbb{R}^N} v(x, t) dx - C_* \right| = |c(t) - C_*| = O(t^{-(N+1)(p-1)}) \quad (4.9)$$

as  $t \rightarrow \infty$ . Furthermore, (2.17) and (4.9) yield

$$\lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j [c(t)P_{t+1} - C_*P_{t+1}]\|_{L^q} = \lim_{t \rightarrow \infty} |c(t) - C_*| = 0 \quad (4.10)$$

for all  $q \in [1, \infty]$  and  $j = 0, 1$ . Put

$$w(x, t) := [e^{t\mathcal{L}}u_0](x) - M(u_0)P_{t+1}(x) \quad (4.11)$$

for all  $(x, t) \in \mathbb{R}^N \times (0, \infty)$ . Since it follows from the semigroup property of  $P_t$  that

$$[e^{t\mathcal{L}}P_1](x) = P_{t+1}(x), \quad (4.12)$$

we have

$$w(x, t) = [e^{t\mathcal{L}}w(0)](x).$$

On the other hand, by (1.5), (4.8) and (4.11) we obtain

$$\int_{\mathbb{R}^N} w(x, 0) dx = 0.$$

Therefore, applying Lemma 2.5 with the aid of (2.18), we see that

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j w(t)\|_{L^q} &= \lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j e^{t\mathcal{L}}w(0)\|_{L^q} \\ &\preceq \lim_{t \rightarrow \infty} \|e^{\frac{t}{2}\mathcal{L}}w(0)\|_{L^1} = 0 \end{aligned} \quad (4.13)$$

for all  $q \in [1, \infty]$  and  $j = 0, 1$ .

Let

$$F(x, t) := |\nabla v(x, t)|^p - M(|\nabla v(t)|^p)P_{t+1}(x). \quad (4.14)$$

Then, by (1.5) and (4.8) we have

$$\int_{\mathbb{R}^N} F(x, t) dx = 0, \quad t \geq 0. \quad (4.15)$$

Since it follows from (4.12) and (4.14) that

$$\begin{aligned} &\int_0^t e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p d\tau - \int_0^t M(|\nabla v(\tau)|^p) d\tau P_{t+1}(x) \\ &= \int_0^t e^{(t-\tau)\mathcal{L}} \{|\nabla v(\tau)|^p - M(|\nabla v(\tau)|^p)P_{\tau+1}\} d\tau = \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) d\tau, \end{aligned}$$

by (1.7), (4.7) and (4.11) we see that

$$\begin{aligned} &v(x, t) - c(t)P_{t+1}(x) \\ &= e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p d\tau - \left[ M(u_0) + \int_0^t M(|\nabla v(\tau)|^p) d\tau \right] P_{t+1}(x) \\ &= w(x, t) + \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) d\tau. \end{aligned}$$

This together with (4.10) and (4.13) implies that

$$\begin{aligned}
& \lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j[v(t) - C_* P_{t+1}]\|_{L^q} \\
&= \lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j[v(t) - c(t)P_{t+1}]\|_{L^q} + \lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|\nabla^j[c(t)P_{t+1} - C_* P_{t+1}]\|_{L^q} \\
&= \lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \left\| \nabla^j \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) d\tau \right\|_{L^q}.
\end{aligned}$$

Therefore, in order to obtain (1.12), it suffices to prove

$$\lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \left\| \nabla^j \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) d\tau \right\|_{L^q} = 0. \quad (4.16)$$

For any  $j \in \{0, 1\}$ , put

$$\begin{aligned}
J_{1,j}(t) &:= \int_{t/2}^t \nabla^j e^{(t-\tau)\mathcal{L}} F(\tau) d\tau, \\
J_{2,j}(t) &:= \int_L^{t/2} \nabla^j e^{(t-\tau)\mathcal{L}} F(\tau) d\tau, \\
J_{3,j}(t) &:= \int_0^L \nabla^j e^{(t-\tau)\mathcal{L}} F(\tau) d\tau,
\end{aligned}$$

for  $t \geq 2L$ , where  $L \geq 1$ . Since it follows from (1.10), (2.17) and (4.14) that

$$\sup_{t>0} (1+t)^{N(1-\frac{1}{q})+N(p-1)+p} \|F(t)\|_{L^q} < \infty, \quad (4.17)$$

by (2.18) we have

$$t^{N(1-\frac{1}{q})} \|J_{1,0}(t)\|_q \preceq t^{N(1-\frac{1}{q})} \int_{t/2}^t \|F(\tau)\|_{L^q} d\tau \preceq t^{-(N+1)(p-1)} = o(1) \quad (4.18)$$

as  $t \rightarrow \infty$ . Furthermore, by (1.9), (2.17) and (4.4) we obtain

$$\begin{aligned}
t^{N(1-\frac{1}{q})+1} \|J_{1,1}(t)\|_{L^q} &\leq t^{N(1-\frac{1}{q})+1} \left[ I_{1,1}(t) + \int_{t/2}^t M(|\nabla v(\tau)|^p) \|\nabla P_{\tau+1}\|_{L^q} d\tau \right] \\
&\preceq t^{N(1-\frac{1}{q})+1} I_{1,1}(t) + \int_{t/2}^t \tau^{-N(p-1)-p} d\tau \\
&\preceq t^{N(1-\frac{1}{q})+1} I_{1,1}(t) + t^{-(N+1)(p-1)} = o(1)
\end{aligned} \quad (4.19)$$

as  $t \rightarrow \infty$ . Moreover, by (2.18) and (4.17) we have

$$\begin{aligned}
t^{N(1-\frac{1}{q})+j} \|J_{2,j}(t)\|_{L^q} &\leq t^{N(1-\frac{1}{q})+j} \int_L^{t/2} (t-\tau)^{-N(1-\frac{1}{q})-j} \|F(\tau)\|_{L^1} d\tau \\
&\preceq \int_L^{t/2} \|F(\tau)\|_{L^1} d\tau \preceq \int_L^{t/2} \tau^{-N(p-1)-p} d\tau \preceq L^{-(N+1)(p-1)}
\end{aligned} \quad (4.20)$$

for all sufficiently large  $t$ . Similarly, we see that

$$\begin{aligned} t^{N(1-\frac{1}{q})+j} \|J_{3,j}(t)\|_{L^q} &\leq t^{N(1-\frac{1}{q})+j} \int_0^L \left\| \nabla^j e^{\frac{(t-\tau)}{2}} \mathcal{L} e^{\frac{(t-\tau)}{2}} \mathcal{L} F(\tau) \right\|_{L^q} d\tau \\ &\preceq \int_0^L \left\| e^{\frac{(t-\tau)}{2}} \mathcal{L} F(\tau) \right\|_{L^1} d\tau \end{aligned} \quad (4.21)$$

for all  $t \geq 2L$ . On the other hand, for any  $L > 0$ , it follows from Lemma 2.5 with (4.15) that

$$\lim_{t \rightarrow \infty} \left\| e^{\frac{(t-\tau)}{2}} \mathcal{L} F(\tau) \right\|_{L^1} = 0 \quad (4.22)$$

for all  $s \in (0, L)$ . Furthermore, by (2.18) we have

$$\sup_{t \geq 2L} \left\| e^{\frac{(t-\tau)}{2}} \mathcal{L} F(\tau) \right\|_{L^1} \leq \|F(\tau)\|_{L^1}. \quad (4.23)$$

Then, applying the Lebesgue dominated convergence theorem with (4.22) and (4.23) to (4.21), we obtain

$$\lim_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \|J_{3,j}(t)\|_{L^q} = 0. \quad (4.24)$$

Therefore, by (4.18), (4.19), (4.20) and (4.24) we see that

$$\limsup_{t \rightarrow \infty} t^{N(1-\frac{1}{q})+j} \left\| \nabla^j \int_0^t e^{(t-\tau)} \mathcal{L} F(\tau) d\tau \right\|_{L^q} \leq C_1 L^{-(N+1)(p-1)}$$

for some constant  $C_1$  independent of  $L$ . Therefore, since  $L$  is arbitrary, by  $p > 1$  we have (4.16), and the proof of the assertion (ii)-(b) of Theorem 1.1 is complete.  $\square$

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## References

- [1] N. Alibaud, C. Imbert and G. Karch, Asymptotic properties of entropy solutions to fractal Burgers equation, *SIAM J. Math. Anal.*, 42 (2010), 354–376.
- [2] L. Amour and M. Ben-Artzi, Global existence and decay for viscous Hamilton-Jacobi equations, *Nonlinear Anal.* 31 (1998), 621–628.
- [3] S. Benachour, G. Karch and Ph. Laurençot, Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations, *J. Math. Pures Appl.* 83 (2004), 1275–1308.
- [4] S. Benachour and Ph. Laurençot, Global solutions to viscous Hamilton-Jacobi equations with irregular initial data, *Comm. Partial Differential Equations* 24 (1999) 1999–2021.



- [5] M. Ben-Artzi and H. Koch, Decay of mass for a semilinear parabolic equation, *Comm. Partial Differential Equations* 24 (1999), 869–881.
- [6] M. Ben-Artzi, Ph. Souplet and F.B. Weissler, The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces, *J. Math. Pures Appl.* 81 (2002), 343–378.
- [7] H. Dong, D. Du and D. Li, Finite time singularities and global well-posedness for fractal Burgers equations, *Indiana Univ. Math. J.*, 58 (2009), 807–822.
- [8] J. Droniou, T. Galloüet and J. Vovelle, Global solution and smoothing effect for a non- local regularization of a hyperbolic equation, *J. Evol. Equ.*, 4 (2003), 479–499.
- [9] J. Droniou and C. Imbert, Fractal first order partial differential equations, *Arch. Rational Mech. Anal.* 182 (2006), 299–331.
- [10] B. Gilding, M. Guedda and R. Kersner, The Cauchy problem for  $u_t = \Delta u + |\nabla u|^q$ , *J. Math. Anal. Appl.* 284 (2003), 733–755.
- [11] C. Imbert, A non-local regularization of first order Hamilton-Jacobi equations, *J. Differential Equations* 211 (2005), 218–246.
- [12] K. Ishige and T. Kawakami, Refined asymptotic profiles a semilinear heat equation, *Math. Ann.*, 353 (2012), 161–192.
- [13] K. Ishige, T. Kawakami and K. Kobayashi, Global solutions for a nonlinear integral equation with a generalized heat kernel, *Discrete Contin. Dyn. Syst. Ser. S.* 7 (2014), 767–783.
- [14] K. Ishige, T. Kawakami and K. Kobayashi, Asymptotics for a nonlinear integral equation with a generalized heat kernel, *J. Evol. Equ.* 14 (2014), 749–777.
- [15] K. Ishige and K. Kobayashi, Convection-diffusion equation with absorption and non-decaying initial data, *J. Differential Equations* 254 (2013), 1247–1268.
- [16] T. Iwabuchi, Global solutions for the critical Burgers equation in the Besov spaces and the large time behavior, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32 (2015), 687–713.
- [17] E. R. Jakobsen and K. H. Karlsen, Continuous dependence estimates for viscosity solutions of integro-PDEs, *J. Differential Equations* 212 (2005), 278–318.
- [18] E. R. Jakobsen and K. H. Karlsen, A "maximum principle for semicontinuous functions" applicable to integro-partial differential equations, *NoDEA Nonlinear Differential Equations Appl.* 13 (2006), 137–165.
- [19] G. Karch and W. A. Woyczyński, Fractal Hamilton-Jacobi-KPZ equations, *Trans. Am. Math. Soc.* 360 (2008), 2423–2442.
- [20] M. Kardar, G. Parisi and Y. C. Zhang, Dynamic scaling of growing interfaces, *Phys. Rev. Lett.* 56 (1986), 889–892.

- [21] A. Kiselev, F. Nazarov and R. Shterenberg, Blow up and regularity for fractal Burgers equation, *Dyn. Partial Differential Equations*, **5** (2008), 211–240.
- [22] H. Kozono, M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, *Comm. Partial Differential Equations*, **19** (1994), no. 5-6, 959–1014.
- [23] J. Krug and H. Spohn, Universality classes for deterministic surface growth, *Phys. Rev. A* **38** (1988), 4271–4283.
- [24] Ph. Laurençot and Ph. Souplet, On the growth of mass for a viscous Hamilton-Jacobi equation, *J. Anal. Math.* **89** (2003), 367–383.
- [25] S. Machihara, T. Ozawa, *Interpolation inequalities in Besov spaces*, *Proc. Amer. Math. Soc.* **131** (2003), no. 5, 1553–1556.
- [26] C. Miao and G. Wu, Global well-posedness of the critical Burgers equation in critical Besov spaces, *J. Differential Equations*, **247** (2009), 1673–1693.
- [27] L. Silvestre, On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion, *Advances in Mathematics* **226** (2011), 2020–2039.
- [28] H. M. Soner, Optimal control with state-space constraint. II, *SIAM J. Control Optim.* **24** (1986), 1110–1122.
- [29] H. Triebel, “Theory of Function Spaces,” Birkhäuser-Verlag, Basel, 1983.